J. Fluid Mech. (1969), vol. 35, part 1, pp. 145–169
Printed in Great Britain


# An analysis of second-order effects on laminar boundary-layer flow

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(Received 18 March 1968)

Treated in the present paper are the second-order effects, i.e. the effect of displacement thickness, longitudinal curvature, external shear and slip, on the two-dimensional laminar boundary-layer flow of incompressible fluid. The analysis is developed in terms of the stream-function co-ordinates proposed for the analysis of the rotational flow by the senior author previously. An inverse problem is set in the co-ordinates plane, and, within the framework of the second-order approximation, the problem is reduced to solving a parabolic partial differential equation for the total head. An implicit finite-difference scheme is then devised to solve the equation, and the practical computations are carried out for the flow along a flat plate or the one around a parabolic body, by making use of an electronic computer.

## 1. Introduction

In connexion with the research on hypersonic low-density flow, attacks on the problem of the so-called second-order effects, i.e. the effect of displacement thickness, longitudinal curvature, transverse curvature (axisymmetric case), external shear and slip, on the laminar boundary-layer flow have been attempted by many authors in recent years. According to analytical views, as set forth soundly by Van Dyke (1962*a*), the problem is reduced to solving asymptotically the full Navier–Stokes equations for large Reynolds number, *Re*, within the framework of the second-order approximation, that is, taking into account the terms up to of order $(Re)^{-\frac{1}{2}}$.

An orthodox approach to the problem could be made by an analytical method to solve the equations through expansion in a series such as the Blasius series in classical boundary-layer theory. In the second-order problem, however, the method has a limitation in its application since the equations under consideration are too complicated by the expansion in series. So some numerical methods to solve the equations for each individual case by making use of electronic computers have been devised by Blottner & Flügge-Lotz (1963), Davis & Flügge-Lotz (1964), Fannelöp & Flügge-Lotz (1966) and others. It seems to the authors, however, that their methods resort too much to the efficiency of electronic computers but then lose something in the analytical simplicity.

In previous papers (Honda 1965, 1966), the senior author proposed a set of co-ordinates called stream function co-ordinates, which are composed of the stream function $\psi$ and a function $\phi$ to be constant on each trajectory perpendicular to the streamlines in the flow field, for the analysis of inviscid rotational flow. The co-ordinate system is now extended to the case of viscous flow and applied to the analysis of the second-order effects on the two-dimensional laminar boundary-layer flow of incompressible fluid. Namely, we start with the Navier–Stokes equations in terms of the stream function co-ordinates and have two sets of equations, inviscid and viscous, within the framework of the second-order approximation. On the assumption that the inviscid solution in the $(\phi, \psi)$-plane is given in advance, the viscous solution in the stretched $(\phi, (Re)^{\frac{1}{2}}\psi)$-plane is formed by introduction of the concept of the matched asymptotic expansions (e.g. Van Dyke 1964*a*). The problem is then reduced to solving a parabolic partial differential equation similar to the Kármán–Millikan (1934) equation in the classical theory. It should be noted here that our approach is an inverse method in view of the fact that the effect of displacement thickness is interpreted as the shift of the body surface on the physical plane.

Next, an implicit finite-difference scheme is devised to solve numerically the differential equation mentioned above. The practical computations to clarify each effect on the boundary-layer flow are then carried out for the flow along a flat plate or the one around a parabolic body, by making use of an electronic computer, and the results obtained are shown in comparison with available ones obtained by the analytical methods.

## 2. Fundamental equations

### 2.1. *Navier–Stokes equations in terms of stream function co-ordinates*

Consider a two-dimensional steady flow of an incompressible viscous fluid. If we write the Navier–Stokes equations in terms of the streamline co-ordinates, denoting the distance measured along a streamline by $s$ and the distance measured perpendicular to it by $n$ in the physical $(x, y)$-plane, the continuity equation is

$$\frac{1}{q}\frac{\partial q}{\partial s}+\frac{\partial \theta}{\partial n} = 0, \tag{2.1}$$

where $q$ is the magnitude of velocity and $\theta$ the direction angle of the velocity to the $x$-axis. The momentum equations are

$$\frac{\partial H}{\partial s} = -\nu\frac{\partial \omega}{\partial n} \tag{2.2}$$

and

$$\frac{\partial H}{\partial n} = -q\omega+\nu\frac{\partial \omega}{\partial s}, \tag{2.3}$$

provided that the total head $H$,

$$H = \tfrac{1}{2}q^2+(p/\rho), \tag{2.4}$$

and the vorticity $\omega$,

$$\omega = q\frac{\partial \theta}{\partial s}-\frac{\partial q}{\partial n}, \tag{2.5}$$

are chosen as the dependent variables. Here $\nu$ is the kinematic viscosity, $p$ the pressure and $\rho$ the density. Equation (2.3) can be rewritten also in an alternative form, i.e.

$$\frac{1}{\rho}\frac{\partial p}{\partial n}+q^2\frac{\partial \theta}{\partial s} = \nu\frac{\partial \omega}{\partial s}. \tag{2.6}$$

Let us now refer all lengths to a typical length $L_r$ and the magnitude of velocity to a characteristic velocity $U_r$, then the kinematic viscosity $\nu$ in the above equations is referred to the reciprocal of the characteristic Reynolds number,

$$Re = U_r L_r/\nu.$$

Further, we shall non-dimensionalize the pressure $p$ with respect to $\rho U_r^2$.

Next, we introduce the stream function co-ordinates which are composed of the stream function $\psi$ and a function $\phi$ defined to be constant on each trajectory perpendicular to the streamlines in the flow field. Transformation of the streamline co-ordinates to the stream function co-ordinates (Honda 1965), which is defined as

$$\frac{1}{q}\frac{\partial}{\partial s} = e^{\Omega}\frac{\partial}{\partial \phi}, \quad \frac{1}{q}\frac{\partial}{\partial n} = \frac{\partial}{\partial \psi} \tag{2.7}$$

and

$$\partial\Omega/\partial\psi = \omega/q^2, \tag{2.8}$$

in (2.2) and (2.3) yields

$$e^{\Omega}(\partial H/\partial\phi) = -\nu(\partial\omega/\partial\psi) \tag{2.9}$$

and

$$\partial H/\partial\psi = -\omega+\nu e^{\Omega}(\partial\omega/\partial\phi). \tag{2.10}$$

Similarly, (2.1) and (2.6) are transformed into

$$e^{\Omega}(\partial q/\partial\phi)+q(\partial\theta/\partial\psi) = 0 \tag{2.11}$$

and

$$(\partial p/\partial\psi)+q^2 e^{\Omega}(\partial\theta/\partial\phi) = \nu e^{\Omega}(\partial\omega/\partial\phi), \tag{2.12}$$

respectively. In addition, the local relation between the stream function co-ordinates $(\phi, \psi)$ and the Cartesian co-ordinates $(x, y)$ in the physical plane can be written as

$$dx+i\,dy = (e^{i\theta}/q)(e^{-\Omega}d\phi+i\,d\psi), \tag{2.13}$$

where $i$ denotes an imaginary unity.

### 2.2. *Inviscid approximation*

If we ignore the terms of order $\nu$ in the non-dimensionalized equations (2.9) to (2.12), these equations are reduced to the so-called Euler-type inviscid equations such as

$$\partial H/\partial\phi = 0; \quad \omega = -(dH/d\psi) \tag{2.14}$$

and

$$e^{\Omega}(\partial p/\partial\phi)-q^2(\partial\theta/\partial\psi) = 0, \tag{2.15a}$$

$$(\partial p/\partial\psi)+q^2 e^{\Omega}(\partial\theta/\partial\phi) = 0, \tag{2.15b}$$

where

$$\frac{\partial\Omega}{\partial\psi} = -\frac{1}{q^2}\frac{dH}{d\psi}. \tag{2.16}$$

In particular, in irrotational flow

$$\omega = 0; \quad \Omega = 0 \tag{2.17}$$

and then $\phi$ reduces to the velocity potential. In this case, the solution of the inviscid equations becomes an exact solution of the full Navier–Stokes equations, but the solution cannot satisfy the boundary condition on the body surface. Namely, the inviscid approximation fails to be valid in the layer surrounding the body surface. In order to prevent this failure, we must introduce an alternative form of approximation to be valid in the layer, that is, the boundary-layer approximation.

### 2.3. *Boundary-layer approximation*

On the analogy of Prandtl boundary-layer theory, we introduce the stretched co-ordinate

$$\psi^* = \psi/\nu^{\frac{1}{2}}. \tag{2.18}$$

Taking into account that the non-dimensionalized vorticity $\omega$ grows to that of the order $1/\nu^{\frac{1}{2}}$, we change the notation so that

$$\omega^* = \nu^{\frac{1}{2}}\omega. \tag{2.19}$$

In terms of the stretched co-ordinates $(\phi, \psi^*)$, the Navier–Stokes equations (2.9) to (2.12) are transformed to

$$e^{\Omega}(\partial H/\partial\phi) = -(\partial\omega^*/\partial\psi^*), \tag{2.20a}$$

$$(\partial H/\partial\psi^*) = -\omega^* + \nu e^{\Omega}(\partial\omega^*/\partial\phi) \tag{2.20b}$$

and

$$\frac{\partial\theta}{\partial\psi^*} = -\nu^{\frac{1}{2}}\frac{e^{\Omega}}{q}\frac{\partial q}{\partial\phi}, \tag{2.21a}$$

$$\frac{\partial p}{\partial\psi^*} = -\nu^{\frac{1}{2}}q^2 e^{\Omega}\frac{\partial\theta}{\partial\phi} + \nu e^{\Omega}\frac{\partial\omega^*}{\partial\phi}. \tag{2.21b}$$

Equation (2.8) becomes

$$\frac{\partial\Omega}{\partial\psi^*} = -\frac{1}{q}\frac{\partial q}{\partial\psi^*} + \nu^{\frac{1}{2}}e^{\Omega}\frac{\partial\theta}{\partial\phi}. \tag{2.22}$$

If we ignore the terms of order of $\nu$ in these equations as in the inviscid approximation, (2.20$a$, $b$) are reduced to

$$\frac{\partial H}{\partial\phi} = e^{-\Omega}\frac{\partial^2 H}{\partial\psi^{*2}} \quad \text{for} \quad H = \tfrac{1}{2}q^2 + p. \tag{2.23}$$

Equation (2.21$a$) gives

$$\theta = \theta^*(\phi) + O(\nu^{\frac{1}{2}}), \tag{2.24}$$

hence

$$(\partial p/\partial\psi^*) = -\nu^{\frac{1}{2}}q^2 e^{\Omega}(d\theta^*/d\phi) \tag{2.25}$$

and

$$\frac{\partial\Omega}{\partial\psi^*} = -\frac{1}{q}\frac{\partial q}{\partial\psi^*} + \nu^{\frac{1}{2}}e^{\Omega}\frac{d\theta^*}{d\phi}. \tag{2.26}$$

It is easily understood that, if the terms of order $\nu^{\frac{1}{2}}$ are ignored further in the above equations, (2.23) reduces to the classical first-order boundary-layer equation introduced by Kármán & Millikan.

## 3. Inverse problem in the $(\phi, \psi^*)$-plane

### 3.1. *Matching of the solutions*

Let us set a problem; assume that an inviscid solution in the $(\phi, \psi)$-plane is given in advance, and then obtain a viscous solution in the $(\phi, \psi^*)$-plane to be matched to the inviscid solution. To solve the problem, we introduce the concept of the matched asymptotic expansions (e.g. Van Dyke 1964*a*). Namely, the viscous solution is determined under the condition that the asymptotic form of the solution for large $\psi^*$ should be matched to the asymptotic form of the inviscid solution for small $\psi$ within the framework of the approximation of order $\nu^{\frac{1}{2}}$.

In the inviscid solution, assume that the body surface is expressed with $\psi = 0$ and the solution can be expanded regularly in the power series of $\psi$ as follows:

$$H = \tfrac{1}{2} + \omega_0 \psi + \ldots \quad (\omega_0 = \text{const.}), \tag{3.1a}$$

$$p = p_0(\phi) + p_1(\phi)\psi + \ldots, \tag{3.1b}$$

$$q^2 = f_0(\phi) + f_1(\phi)\psi + \ldots, \tag{3.1c}$$

$$\theta = \theta_0(\phi) + \theta_1(\phi)\psi + \ldots \tag{3.1d}$$

and

$$\Omega = \Omega_0(\phi) + \Omega_1(\phi)\psi + \ldots. \tag{3.1e}$$

Here we set the reference pressure equal to zero and have

$$\tfrac{1}{2}f_0(\phi) + p_0(\phi) = \tfrac{1}{2} \tag{3.2a}$$

and

$$\tfrac{1}{2}f_1(\phi) + p_1(\phi) = \omega_0. \tag{3.2b}$$

In case of the inviscid rotational flow, for convenience, we change the notation so that

$$\Omega - \Omega_0(\phi) \to \Omega \quad \text{and} \quad \exp[-\Omega_0(\phi)]\,d\phi \to d\phi. \tag{3.3}$$

It is evident that both the inviscid and the viscous equations are not altered in the form by this transformation, with an exception that (3.1*e*) is rewritten as

$$\Omega = \Omega_1(\phi)\psi + \ldots. \tag{3.4}$$

Substitution of (3.1*a*–*d*) and (3.4) into (2.15*a*, *b*) and (2.16) gives the following relations between the values at $\psi = 0$ and their derivatives with respect to $\psi$,

$$p_1(\phi) = -f_0(\phi)\theta_0'(\phi), \tag{3.5a}$$

$$f_1(\phi) = 2f_0(\phi)\theta_0'(\phi) + 2\omega_0, \tag{3.5b}$$

$$\theta_1(\phi) = -\tfrac{1}{2}f_0'(\phi)/f_0(\phi) \tag{3.5c}$$

and

$$\Omega_1(\phi) = -\omega_0/f_0(\phi), \tag{3.5d}$$

where the prime denotes the differentiation with respect to $\phi$. In terms of the stretched co-ordinates, therefore, the inviscid solution is expressed as

$$H = \tfrac{1}{2} + \nu^{\frac{1}{2}}\omega_0\psi^*, \tag{3.6a}$$

$$p = p_0 - \nu^{\frac{1}{2}} f_0 \theta_0' \psi^*, \tag{3.6b}$$

$$q^2 = f_0 + 2\nu^{\frac{1}{2}}(f_0\theta_0' + \omega_0)\psi^*, \tag{3.6c}$$

$$\theta = \theta_0 - \frac{\nu^{\frac{1}{2}}}{2}\frac{f_0'}{f_0}\psi^* \tag{3.6d}$$

and

$$\Omega = -\nu^{\frac{1}{2}}(\omega_0/f_0)\psi^*, \tag{3.6e}$$

within the approximation of order $\nu^{\frac{1}{2}}$.

In the matching, we have first

$$\theta^*(\phi) = \theta_0(\phi), \tag{3.7}$$

in comparison with matching (2.24) and (3.6$d$). If we denote $q^2$ in the inviscid solution by $U^2$, i.e.

$$U^2 = f_0 + 2\nu^{\frac{1}{2}}(f_0\theta_0' + \omega_0)\psi^*, \tag{3.8}$$

integration of (2.26) gives

$$e^{\Omega} = (U/q) + O(\nu^{\frac{1}{2}}). \tag{3.9}$$

Substituting it into the term of order $\nu^{\frac{1}{2}}$ and integrating the equation again, we get

$$\Omega = -\log q + \nu^{\frac{1}{2}}\theta_0' \int_0^{\psi^*} \frac{U}{q}\, d\psi^* + C(\phi). \tag{3.10}$$

Here an arbitrary function of $\phi$, $C(\phi)$, should be determined on the matching principle. Since $q \to U$ as $\psi^* \to \infty$, the principle yields

$$-\nu^{\frac{1}{2}}\frac{\omega_0}{f_0}\psi^* = -\tfrac{1}{2}\log U^2 + \nu^{\frac{1}{2}}\theta_0'\psi^* + \nu^{\frac{1}{2}}\theta_0'\int_0^{\infty}\left(\frac{U}{q}-1\right)d\psi^* + C(\phi), \tag{3.11}$$

in view of (3.6$e$). Taking into account that

$$\tfrac{1}{2}\log U^2 = \tfrac{1}{2}\log f_0 + \nu^{\frac{1}{2}}(\theta_0' + \omega_0/f_0)\psi^*, \tag{3.12}$$

we have

$$C(\phi) = \tfrac{1}{2}\log f_0 - \nu^{\frac{1}{2}}\theta_0'\int_0^{\infty}\left(\frac{U}{q}-1\right)d\psi^*. \tag{3.13}$$

Consequently, the solution of $\Omega$ is obtained in the form

$$\Omega = -\log\frac{q}{U} - \nu^{\frac{1}{2}}\frac{\omega_0}{f_0}\psi^* - \nu^{\frac{1}{2}}\theta_0'\int_{\psi^*}^{\infty}\left(\frac{U}{q}-1\right)d\psi^* \tag{3.14}$$

or

$$e^{-\Omega} = \frac{q}{U}\exp\left[\nu^{\frac{1}{2}}\frac{\omega_0}{f_0}\psi^* + \nu^{\frac{1}{2}}\theta_0'\int_{\psi^*}^{\infty}\left(\frac{U}{q}-1\right)d\psi^*\right]. \tag{3.15}$$

In the similar way, we get

$$p = \tfrac{1}{2}(1-U^2) + \nu^{\frac{1}{2}}\omega_0\psi^* - \nu^{\frac{1}{2}}f_0\theta_0'\int_{\psi^*}^{\infty}\left(1-\frac{q}{U}\right)d\psi^*, \tag{3.16}$$

which results in the following relation between $H$ and $q^2$;

$$1 - 2(H - \nu^{\frac{1}{2}}\omega_0\psi^*) = U^2 - q^2 + 2\nu^{\frac{1}{2}}f_0\theta_0'\int_{\psi^*}^{\infty}\left(1-\frac{q}{U}\right)d\psi^*. \tag{3.17}$$

The flow-direction angle $\theta$ can be obtained also under the same procedure. Integration of (2.21$a$) gives

$$\theta = \theta_0 - \frac{\nu^{\frac{1}{2}}}{2}\frac{f_0'}{f_0}\psi^* + \nu^{\frac{1}{2}}\int_{\psi^*}^{\infty}\left[\frac{U}{q^2}\frac{\partial q}{\partial\phi} - \frac{1}{U}\frac{\partial U}{\partial\phi}\right]d\psi^*, \tag{3.18}$$

in which we can rewrite

$$\frac{U}{q^2}\frac{\partial q}{\partial\phi} - \frac{1}{U}\frac{\partial U}{\partial\phi} = -U\frac{\partial}{\partial\phi}\left[\frac{1}{q}-\frac{1}{U}\right]. \tag{3.19}$$

Hence, we have

$$\theta = \theta_0 - \frac{\nu^{\frac{1}{2}}}{2}\frac{f_0'}{f_0}\psi^* - \nu^{\frac{1}{2}}U\frac{\partial}{\partial\phi}\left[\int_{\psi^*}^{\infty}\left(\frac{1}{q}-\frac{1}{U}\right)d\psi^*\right], \tag{3.20}$$

within the approximation of order $\nu^{\frac{1}{2}}$.

### 3.2. *Displacement thickness*

As mentioned in § 1, our problem is an inverse one because the body shape derived from the viscous solution in the $(\phi, \psi^*)$-plane becomes different from that assumed for the inviscid solution in advance.† The distance along a $\phi$ = const. line in the viscous solution is

$$n = \nu^{\frac{1}{2}} \int \frac{d\psi^*}{q}. \tag{3.21}$$

On the other hand, the distance in the inviscid solution is

$$n = \nu^{\frac{1}{2}} \int \frac{d\psi^*}{U}. \tag{3.22}$$

Therefore, the body surface corresponding to $\psi^* = 0$ is shifted with

$$\delta^* = \nu^{\frac{1}{2}} \int_0^\infty \left(\frac{1}{q} - \frac{1}{U}\right) d\psi^* \tag{3.23}$$

from the original position.

In the usual notation, i.e. $\nu^{\frac{1}{2}} d\psi^* = q\, dn$, the quantity defined above is expressed as

$$\delta^* = \int_0^\infty \left(1 - \frac{q}{U}\right) dn, \tag{3.24}$$

the displacement thickness.‡ The body shape in the physical $(x, y)$-plane becomes

$$x_b = x_0 + \delta^* \sin\theta_0, \tag{3.25a}$$

$$y_b = y_0 - \delta^* \cos\theta_0, \tag{3.25b}$$

within the approximation of order $\nu^{\frac{1}{2}}$. Here the co-ordinates $(x_0, y_0)$ denote the body shape assumed for the inviscid solution. The local slope of the body surface $\theta_b$ is derived as

$$\theta_b = \theta_0 - f_0^{\frac{1}{2}}(d\delta^*/d\phi) \tag{3.26}$$

from (3.20).

## 4. Numerical method of solution

The partial differential equation for the total head $H$, equation (2.23), is a nonlinear parabolic equation, regardless of the first- and second-order approximations. It means that the mathematical treatment of the equation is quite similar up to the second-order approximation. For simplicity, let us describe a computation scheme to solve the first-order equation.

† The body surface assumed for the inviscid solution in advance corresponds to the displacement surface in the direct problem (Catherall & Mangler 1966).

‡ The present definition of the displacement thickness is the same as the one by Murphy (1965), while Schultz-Grunow & Breuer (1965, p. 377) define the thickness as

$$\delta^* = \int_0^\infty \left(1 - \frac{q}{U}\right) \frac{U}{U_0}\, dn \quad (U_0^2 = f_0),$$

and Werle & Davis (1966) define it as

$$\int_0^{\delta^*} U\, dn = \int_0^\infty (U - q)\, dn.$$

The first-order equation which is obtained by setting $\nu^{\frac{1}{2}} = 0$ in equations (2.25), (3.15) and (3.17) is certainly the Kármán–Millikan equation:

$$\frac{\partial H}{\partial \phi} = \frac{q}{U_0}\frac{\partial^2 H}{\partial \psi^{*2}}, \tag{4.1a}$$

$$\left(\frac{q}{U_0}\right)^2 = 1 - \frac{1-2H}{U_0^2}, \tag{4.1b}$$

where

$$U_0^2 = f_0(\phi). \tag{4.1c}$$

Transformation of variables

$$h = \frac{1-2H}{f_0}, \quad t = \frac{\psi^*}{\surd(2\phi)} \tag{4.2}$$

in the above equations yields

$$u\frac{\partial^2 h}{\partial t^2} + t\frac{\partial h}{\partial t} - \beta h = 2\phi\frac{\partial h}{\partial \phi}, \tag{4.3a}$$

$$\beta(\phi) = \frac{2\phi}{f_0}\frac{df_0}{d\phi} \tag{4.3b}$$

and

$$u^2 = 1 - h, \tag{4.4}$$

where we change the notation so that

$$q/U_0 \rightarrow u. \tag{4.5}$$

The boundary conditions to be satisfied by $h$ are

$$h = 1 \quad \text{at} \quad t = 0 \quad \text{(no-slip case)}, \tag{4.6a}$$

$$h \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \tag{4.6b}$$

As is well known, (4.3$a$) has a defect for the numerical calculation because there is a singularity at $t = 0$, written in such a form as

$$h = 1 - at + \tfrac{4}{3}\beta a^{-\frac{1}{2}} t^{\frac{3}{2}} + O(t^2), \tag{4.7a}$$

$$\partial^2 h/\partial t^2 = \beta a^{-\frac{1}{2}} t^{-\frac{1}{2}} + O(1). \tag{4.7b}$$

In order to obviate this defect, we change the independent variable $t$ to $\eta$,

$$t = \tfrac{1}{2}\eta^2, \tag{4.8}$$

then

$$\frac{\partial^2 h}{\partial \eta^2} - \left[\frac{1}{\eta} - \frac{1}{2}\frac{\eta^3}{u}\right]\frac{\partial h}{\partial \eta} - \beta\frac{\eta^2}{u}h = 2\phi\frac{\eta^2}{u}\frac{\partial h}{\partial \phi}. \tag{4.9}$$

To solve the equation numerically, we adopt an implicit finite-difference scheme as described in the following.

Consider a general problem to obtain the value of $h$ on a line $\phi = \phi$ from the value on a line $\phi = \phi - \Delta\phi$ in the $(\phi, \eta)$-plane. (In practice, a function $\xi$ suitable to each problem will be chosen in place of $\phi$, but the treatment is entirely the same.) First, by replacing the $\phi$-derivative appearing on the right-hand side in (4.9) by the finite difference as

$$\frac{\partial h}{\partial \phi} = \frac{1}{\Delta\phi}[h(\phi) - h(\phi - \Delta\phi)], \tag{4.10}$$

we have
$$\frac{\partial^2 h}{\partial \eta^2}-\left[\frac{1}{\eta}-\frac{1}{2}\frac{\eta^3}{u}\right]\frac{\partial h}{\partial \eta}-\left(\beta+\frac{2\phi}{\Delta\phi}\right)\frac{\eta^2}{u}h$$
$$=-\frac{2\phi}{\Delta\phi}\frac{\eta^2}{u}h(\phi-\Delta\phi). \tag{4.11}$$

There are two courses in solving the above equation: one is to solve it as an ordinary differential equation with respect to $\eta$ by using the Runge–Kutta or other method (e.g. Smith & Clutter 1963, 1965), the other to solve it through the conversion of the differential equation to a difference equation (e.g. Blottner & Flügge-Lotz 1963; Krause 1967). We take the latter course because the problem under consideration is a two-point boundary-value problem and the latter is preferable to the former for the problem.

Let us replace the boundary condition (4.6*b*) by the condition
$$h = 0 \quad \text{at} \quad \eta = \eta_N, \tag{4.12}$$
choosing a definite value $\eta_N$ suitably. We next divide $\eta_N$ into $N$ by an equal interval $\Delta\eta$ and denote the value of $h$ at $\eta = m\Delta\eta$ by $h_m$. By replacing the first and second derivatives with respect to $\eta$ on the left-hand side in (4.11) by finite differences through the Lagrange 5-point difference formula, we can convert the equation to a set of quasi-linear simultaneous equations for $h_m$ ($1 \leqslant m \leqslant N-1$) which can be expressed in the form
$$\sum_{n=0}^{4} a_{1,n}h_n = b_1, \tag{4.13a}$$
$$\sum_{n=m-2}^{m+2} a_{m,n}h_n = b_m \quad (2 \leqslant m \leqslant N-2), \tag{4.13b}$$
$$\sum_{n=N-4}^{N} a_{N-1,n}h_n = b_{N-1}, \tag{4.13c}$$
where
$$h_0 = 1, \quad h_N = 0. \tag{4.14}$$
The coefficients in the above equations include
$$u_m = [1-h_m]^{\frac{1}{2}} \tag{4.15}$$
but they could be calculated by the usual iteration process. The practical calculations, in which
$$\Delta\eta = 0{\cdot}1, \quad N = 30; \quad \eta_N = 3{\cdot}0 \tag{4.16}$$
are taken, were carried out by making use of the electronic computer, HITAC-5020 E, in the Computation Centre, University of Tokyo. In order to exhibit the accuracy of our numerical method, let us first describe some results obtained for the first-order boundary-layer problem before the second-order problem.

## 5. First-order boundary-layer problem

### 5.1. *Howarth velocity distribution*

Assume that the velocity distribution along a flat plate is described with
$$U_0(s) = 1-s \tag{5.1}$$

in the inviscid solution, where $s$ is the distance measured from the leading edge of the plate. The function $\phi$ is

$$\phi = s - \tfrac{1}{2}s^2, \tag{5.2}$$

hence

$$U_0^2(\phi) = 1 - 2\phi \tag{5.3}$$

and $\beta(\phi)$ defined by (4.3$b$) becomes

$$\beta(\phi) = -4\phi/(1-2\phi). \tag{5.4}$$

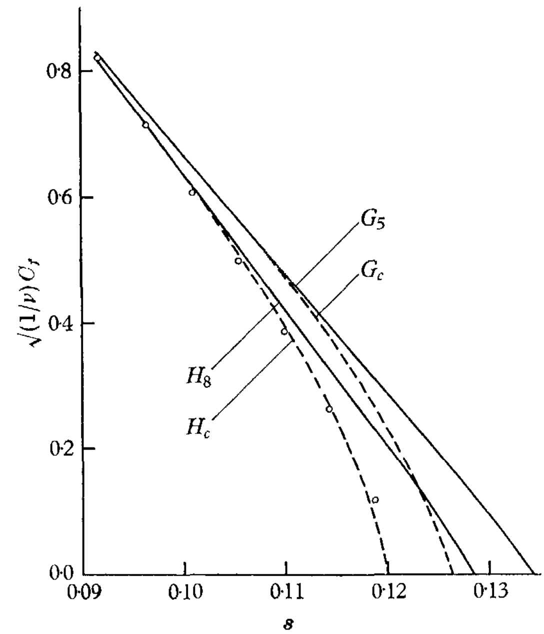


FIGURE 1. Distribution of the skin-friction coefficient for the Howarth velocity distribution. ○, present method; $H_8$, Howarth (1938) 8-term series solution; $G_5$, Görtler (1957) 5-term series solution; $H_c$, Howarth (1938) numerical solution; $G_c$, Görtler (1957) numerical solution.

If we change the independent variable $\phi$ in (4.9) to $\xi$,

$$\xi = -\beta(\phi) = 4\phi/(1-2\phi), \tag{5.5}$$

for convenience in computation, we have

$$\frac{\partial^2 h}{\partial \eta^2} - \left[\frac{1}{\eta} - \frac{1}{2}\frac{\eta^3}{u}\right]\frac{\partial h}{\partial \eta} + \xi\frac{\eta^2}{u}h = \xi(2+\xi)\frac{\eta^2}{u}\frac{\partial h}{\partial \xi}. \tag{5.6}$$

The solution at $\xi = 0$ is a similar solution corresponding to the Blasius solution, i.e. the solution of the ordinary differential equation

$$\frac{d^2 h}{d\eta^2} - \left[\frac{1}{\eta} - \frac{1}{2}\frac{\eta^3}{u}\right]\frac{dh}{d\eta} = 0. \tag{5.7}$$

In solving the equation, we start with $u = 1$ (Kármán–Millikan approximate method) and carry out 7-cycle iteration. The error in the value of $h$ obtained thus seems to be confined to $1 \times 10^{-5}$ in comparison with the value derived from the Blasius solution tabulated accurately up to the sixth decimal (Rosenhead 1963).

The step-by-step computation, taking $\Delta\xi = 0{\cdot}025$, proceeds then downstream with 4-cycle iteration in each step. The skin friction $\tau$ is obtained by

$$\frac{\tau}{\rho} = \nu\frac{\partial q}{\partial n} = \frac{\nu^{\frac{1}{2}}}{(2\phi)^{\frac{1}{2}}} U_0^{\frac{3}{2}} \left(\frac{\partial u}{\partial \eta}\right)^2_{\eta=0}, \tag{5.8}$$

in which the value of the $\eta$-derivative can be estimated through the 5-point difference formula. In our computation, the value becomes minus at $\xi = 0{\cdot}600$, that is, the computation gives the flow-separation point between

$$\xi = 0{\cdot}575 \sim 0{\cdot}600; \quad s = 0{\cdot}1187 \sim 0{\cdot}1229.$$

Figure 1 shows the distribution of the skin-friction coefficient,

$$C_f = 2\tau/\rho, \tag{5.9}$$

emphasizing the neighbourhood of the separation point. The figure includes Howarth (1938) 8-term series solution, Görtler (1957) 5-term series solution and their numerical solutions obtained for the region where the convergence of their series solution becomes slow. As seen in the figure, our result coincides very nearly with the Howarth numerical solution.

### 5.2. *Flow around a parabolic body*

Consider first an inviscid flow around a parabolic body with nose radius of unity placed in a uniform stream of the velocity of unity. The solution in the $(\phi, \psi)$-plane, i.e. the complex potential plane, is derived from the relations

$$x + iy = \tfrac{1}{2} - \tfrac{1}{2}(1+\zeta)^2 \tag{5.10}$$

and

$$\phi + i\psi = -\tfrac{1}{2}\zeta^2, \tag{5.11}$$

by introducing an auxiliary complex variable $\zeta$. Along the surface, putting $\psi = 0$, we have

$$x_0 = \phi, \quad y_0 = \sqrt{(2\phi)} \tag{5.12}$$

and

$$U_0^2 = \frac{2\phi}{1+2\phi}, \quad \theta_0 = \tan^{-1}\frac{1}{\sqrt{(2\phi)}}. \tag{5.13}$$

Hence, $\beta(\phi)$ in (4.9) becomes

$$\beta(\phi) = 2/(1+2\phi). \tag{5.14}$$

Change of the variable $\phi$ to $\xi$ defined by

$$\xi = 2\phi/(1+2\phi) \tag{5.15}$$

in (4.9) results in

$$\frac{\partial^2 h}{\partial \eta^2} - \left[\frac{1}{\eta} - \frac{1}{2}\frac{\eta^3}{u}\right]\frac{\partial h}{\partial \eta} - 2(1-\xi)\frac{\eta^2}{u}h = 2\xi(1-\xi)\frac{\eta^2}{u}\frac{\partial h}{\partial \xi}. \tag{5.16}$$

Starting with the solution at $\xi = 0$,

$$\frac{d^2 h}{d\eta^2} - \left[\frac{1}{\eta} - \frac{1}{2}\frac{\eta^3}{u}\right]\frac{dh}{d\eta} - 2\frac{\eta^2}{u}h = 0, \tag{5.17}$$

which corresponds to the similar solution for the stagnation-point flow, we carry out the step-by-step computation up to $\xi = 1$ with $\Delta\xi = 0{\cdot}05$, under the same procedure as employed in the previous problem. The value of the skin-friction coefficient is tabulated in table 1.

In this problem, both the Blasius series in powers of the distance from the nose along the body surface and the Görtler series in powers of the co-ordinate $\phi$ have the singularity which limits the convergence of the series. In order to improve the convergence, applying the Euler transformation, Van Dyke (1964*b*)

| $\xi$ | Present | Van Dyke (1964*b*) | Görtler (1957) |
|---|---|---|---|
| 0·00 | 0·0000 | 0·0000 | 0·0000 |
| 0·05 | 0·0853 | 0·0854 | 0·0854 |
| 0·10 | 0·1669 | 0·1672 | 0·1672 |
| 0·15 | 0·2449 | 0·2452 | 0·2452 |
| 0·20 | 0·3190 | 0·3194 | 0·3194 |
| 0·25 | 0·3891 | 0·3895 | 0·3894 |
| 0·30 | 0·4550 | 0·4554 | 0·4549 |
| 0·35 | 0·5166 | 0·5168 | 0·5149 |
| 0·40 | 0·5734 | 0·5737 | 0·5662 |
| 0·45 | 0·6252 | 0·6256 | 0·5993 |
| 0·50 | 0·6721 | 0·6724 | 0·5837 |
| 0·55 | 0·7132 | 0·7137 | 0·4200 |
| 0·60 | 0·7483 | 0·7492 | −0·2047 |
| 0·65 | 0·7769 | 0·7784 | |
| 0·70 | 0·7982 | 0·8010 | |
| 0·75 | 0·8114 | 0·8165 | |
| 0·80 | 0·8153 | 0·8242 | |
| 0·85 | 0·8076 | 0·8237 | |
| 0·90 | 0·7860 | 0·8141 | |
| 0·95 | 0·7450 | 0·7949 | |
| 1·00 | 0·6664 | 0·7650 | |

TABLE 1. Values of $(\phi/\nu)^{\frac{1}{2}}C_f$.

modifies the Blasius series into a series in powers of the variable $\xi$. The values derived from the modified Blasius 6-term series solution and the Görtler 6-term series solution are included in table 1. The accurate value at $\xi = 1$, i.e. $\phi \to \infty$, should be the value for the flat-plate flow, that is 0·6641.

## 6. Curvature effect

Let us obtain the viscous solution, taking into account the terms up to of order $\nu^{\frac{1}{2}}$, to be matched to the potential flow around the parabolic body. As seen in (3.15), (3.16) and (3.17), the terms of order $\nu^{\frac{1}{2}}$ manifest the curvature effect on the boundary-layer flow through

$$\theta_0'(\phi) = -1/U_0 R, \tag{6.1}$$

where $R$ is the local radius of curvature of the parabolic body assumed for the inviscid solution. By using the variable $\xi$ defined by (5.15), $\theta_0'(\phi)$ is rewritten as

$$\theta_0'(\phi) = -\frac{1}{\sqrt{(2\phi)(1+2\phi)}} = -\frac{1-\xi}{\sqrt{(2\phi)}}. \tag{6.2}$$

Substituting it into (3.15) and (3.17), we have

$$e^{-\Omega} = \frac{q}{U}\left[1-\nu^{\frac{1}{2}}(1-\xi)\int_{\eta}^{\infty}\left(\frac{U}{q}-1\right)\eta\, d\eta\right] \tag{6.3}$$

and

$$\left(\frac{q}{U}\right)^2 = 1-\frac{h+2\nu^{\frac{1}{2}}(1-\xi)\int_{\eta}^{\infty}\left(1-\frac{q}{U}\right)\eta\, d\eta}{1-\nu^{\frac{1}{2}}(1-\xi)\eta^2}, \tag{6.4}$$

within the approximation of order $\nu^{\frac{1}{2}}$. The differential equation for $h$ is the same as the first-order equation (5.16), except that $u$ included in the coefficients is replaced by $e^{-\Omega}$ defined above. The boundary condition at $\eta = 0$, however, should be modified as

$$h = 1-2\nu^{\frac{1}{2}}(1-\xi)\int_{0}^{\infty}\left(1-\frac{q}{U}\right)\eta\, d\eta \quad \text{at} \quad \eta = 0. \tag{6.5}$$

Analytically speaking, the second-order effects are to be linear with the parameter $\nu^{\frac{1}{2}}$. In practice, we are obliged to carry out the calculations for a series of small values of $\nu^{\frac{1}{2}}$, in order to ensure the numerical results. However, it should be kept in mind that no significance is to be attached to any nonlinear variation with $\nu^{\frac{1}{2}}$ thus found, because it might be involved in unnecessary third-order effects.

Taking the parameter $\nu^{\frac{1}{2}} = 0{\cdot}02$, $0{\cdot}04$, $0{\cdot}06$, $0{\cdot}08$ and $0{\cdot}10$, we carry out the step-by-step computations under the same procedure as in the first-order problem. Here the integrals with respect to $\eta$ appearing in (6.3), (6.4) and (6.5) are calculated by adopting the trapezoid rule in such a form as

$$\int_{\eta_m}^{\infty} F(\eta)\, d\eta = \Delta\eta\left[\sum_{n=N-1}^{m+1} F(\eta_n)+\tfrac{1}{2}F(\eta_m)\right] \quad (F(\eta_N) = 0), \tag{6.6}$$

in which the value of $F(\eta)$ takes the value obtained in the cycle just before.

Figure 2 shows the velocity profile in the boundary layer at $\xi = 0$, i.e. stagnation point, where the distance $n$ is calculated by applying the Simpson rule to the following relation

$$n = \int_{0}^{\psi}\frac{d\psi}{q} = \frac{\sqrt{(2\nu\phi)}}{U_0}\int_{0}^{\eta}\frac{U_0}{q}\eta\, d\eta$$

$$= \nu^{\frac{1}{2}}(1+2\phi)^{\frac{1}{2}}\int_{0}^{\eta}\frac{U}{q}\left[1-\nu^{\frac{1}{2}}(1-\xi)\eta^2\right]^{-\frac{1}{2}}\eta\, d\eta. \tag{6.7}$$

In appearance, it may be concluded that the effect of curvature on the skin friction is rather small. On the other hand, Van Dyke (1962*b*) has obtained the result of

$$\tau/\rho = \nu^{\frac{1}{2}}s[1{\cdot}2326-1{\cdot}9133\nu^{\frac{1}{2}}] \tag{6.8}$$

for the curvature effect on the stagnation-point flow, where $s$ is the distance measured from the stagnation point along the surface. In our calculation, the skin friction is obtained from

$$\frac{\tau}{\rho} = \frac{\sqrt{(2\nu\phi)}}{1+2\phi}\left[\frac{\partial}{\partial\eta}\left(\frac{q}{U}\right)\right]^2_{\eta=0} \tag{6.9}$$

and the limit expression for $\phi \to 0$ is

$$\frac{\tau}{\rho} = \sqrt{(2\nu\phi)}\left[\frac{\partial}{\partial\eta}\left(\frac{q}{U}\right)\right]^2_{\xi,\,\eta=0}. \tag{6.10}$$

The apparent disagreement with Van Dyke's result has, in fact, a very simple explanation. It is the fact that, in our problem, the body surface is shifted from the original position and then the nose radius of the body is affected by the parameter $\nu^{\frac{1}{2}}$.

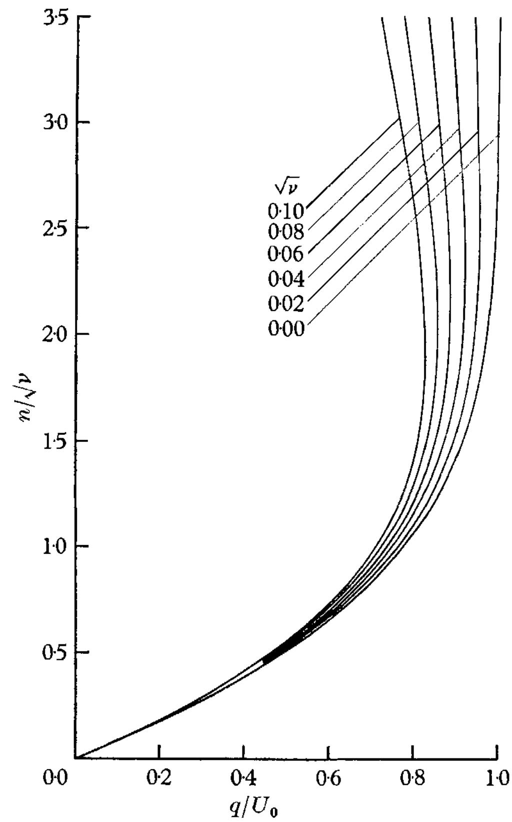


FIGURE 2. Curvature effect on the velocity profile in the boundary layer at the stagnation point.

The co-ordinates of the body $(x_b, y_b)$ in the physical plane are

$$x_b = \phi + \frac{\delta^*}{\sqrt{(1+2\phi)}}, \tag{6.11a}$$

$$y_b = \sqrt{(2\phi)}\left[1 - \frac{\delta^*}{\sqrt{(1+2\phi)}}\right], \tag{6.11b}$$

in view of equation (3.25). On the other hand, the distance $s$ along the surface can be derived as follows.

Equation (6.3) gives

$$\left[\frac{e^{-\Omega}}{q}\right]_{\eta=0} = \frac{1}{U_0}\left[1 - \frac{\delta^*}{(1+2\phi)^{\frac{3}{2}}}\right], \tag{6.12}$$

within the approximation of order $\nu^{\frac{1}{2}}$. Since

$$s = \int_0^\phi \left[\frac{e^{-\Omega}}{q}\right]_{\eta=0} d\phi \tag{6.13}$$

along the surface, we get

$$s = (1-\delta_s^*)s_0 + \int_0^\phi \left[\delta_s^* - \frac{\delta^*}{(1+2\phi)^{\frac{3}{2}}}\right]\frac{d\phi}{U_0}, \tag{6.14}$$

where $\delta_s^*$ denotes the displacement thickness at $\phi = 0$ and $s_0$ the distance in the inviscid solution, i.e.

$$s_0 = \int_0^\phi \frac{d\phi}{U_0} = \frac{\sqrt{(2\phi)}}{2}\left[\sqrt{(1+2\phi)} + \frac{\sinh^{-1}\sqrt{(2\phi)}}{\sqrt{(2\phi)}}\right]. \tag{6.15}$$

| | | | | | $T$ | |
|---|---|---|---|---|---|---|
| $\sqrt{\nu}$ | $\left[\frac{\partial}{\partial\eta}\left(\frac{q}{U}\right)\right]^2_{\xi,\eta=0}$ | $\frac{\delta_s^*}{\sqrt{\nu}}$ | $L$ | $\frac{L}{\nu}$ | Present | Van Dyke (1962*b*) |
| 0·00 | 1·2302 | 0·6480 | 1·0000 | $\infty$ | 1·2302 | 1·2326 |
| 0·02 | 1·2225 | 0·6589 | 0·9738 | 2435 | 1·1905 | 1·1938 |
| 0·04 | 1·2147 | 0·6709 | 0·9470 | 591·9 | 1·1504 | 1·1540 |
| 0·06 | 1·2065 | 0·6842 | 0·9196 | 255·4 | 1·1095 | 1·1129 |
| 0·08 | 1·1981 | 0·6987 | 0·8913 | 139·3 | 1·0679 | 1·0705 |
| 0·10 | 1·1891 | 0·7144 | 0·8622 | 86·2 | 1·0253 | 1·0266 |

TABLE 2. Values of $T = \frac{\tau}{\rho}\Big/\left[\left(\frac{\nu}{L}\right)^{\frac{1}{2}}\frac{s}{L}\right]$.

The limiting process for $\phi \to 0$ in (6.11), (6.14) and (6.15) gives

$$x_b = \delta_s^* + \frac{1}{2}\frac{y_b^2}{(1-\delta_s^*)^2} \tag{6.16}$$

and

$$s = (1-\delta_s^*)\sqrt{(2\phi)}. \tag{6.17}$$

Equation (6.16) manifests that the nose radius of the body $L$ becomes

$$L = (1-\delta_s^*)^2. \tag{6.18}$$

Provided that the nose radius $L$ is chosen as the typical length, we should take $L/\nu$ as the characteristic Reynolds number and $s/L$ as the non-dimensionalized length. From this point of view, we can rewrite (6.10) in the form

$$\frac{\tau}{\rho} = \left(\frac{\nu}{L}\right)^{\frac{1}{2}}\frac{s}{L}T, \quad T = L\left[\frac{\partial}{\partial\eta}\left(\frac{q}{U}\right)\right]^2_{\xi,\eta=0} \tag{6.19}$$

Table 2 shows the numerical values appearing in the above equation in comparison with Van Dyke's result, written as

$$T = 1{\cdot}2326 - 1{\cdot}9133(\nu/L)^{\frac{1}{2}} \tag{6.20}$$

in the present notations.

The body shape for $\nu^{\frac{1}{2}} = 0{\cdot}10$ is shown in figure 3, where the chained line is a parabola with the nose radius $L$ defined by (6.18). Figure 4 shows the distribution of the skin-friction coefficient calculated by (6.9) and of the pressure coefficient

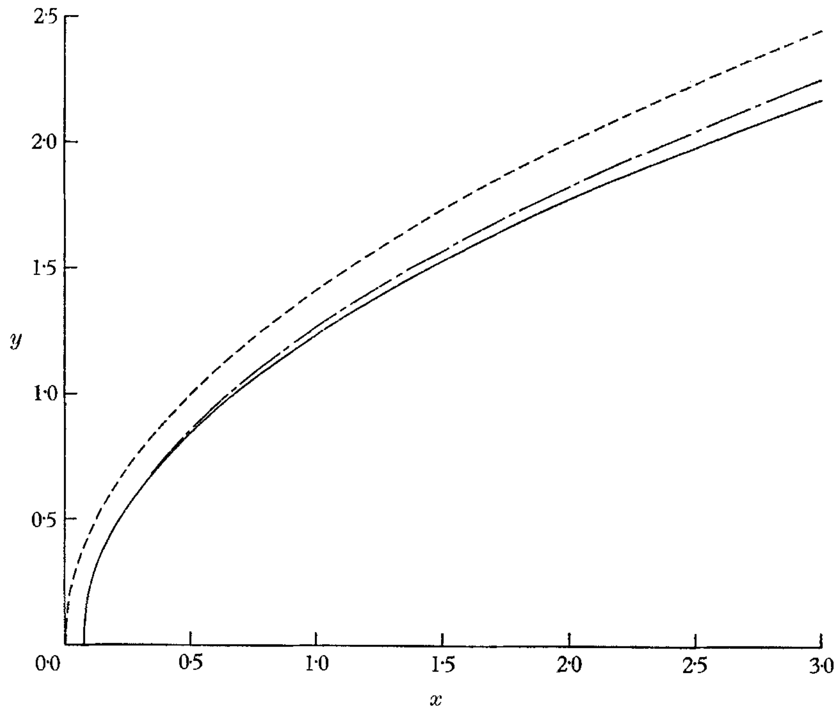


FIGURE 3. Body shape for $\nu^{\frac{1}{2}} = 0{\cdot}10$. ----, body shape assumed for the inviscid solution; — — —, parabola with the nose radius $L = (1-\delta_s^*)^2$.

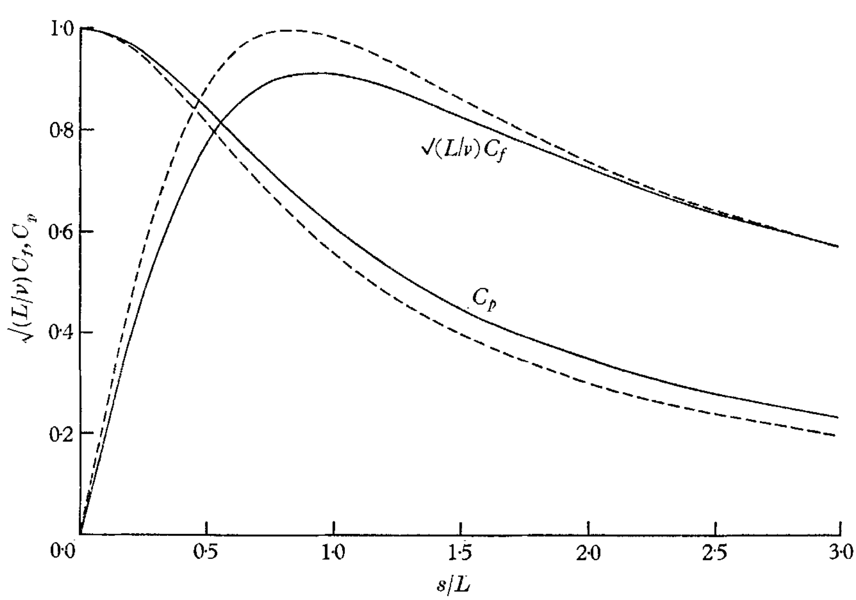


FIGURE 4. Distribution of the skin-friction coefficient and of the pressure coefficient for $\nu^{\frac{1}{2}} = 0{\cdot}10$. ----, those for $\nu^{\frac{1}{2}} = 0$.

with abscissa of the non-dimensionalized distance $s/L$. Here the pressure coefficient $C_p$ is obtained through

$$C_p = \frac{1}{1+2\phi}\left[1+2\nu^{\frac{1}{2}}\xi\int_0^\infty\left(1-\frac{q}{U}\right)\eta\,d\eta\right], \qquad (6.21)$$

to be derived from (3.16).

## 7. External-shear effect

Consider a boundary-layer flow to be matched to an inviscid shear flow on a semi-infinite flat plate. Here we assume that the shear flow is described with

$$U = 1+\omega_0 y \quad (\omega_0 > 0) \qquad (7.1)$$

in the physical plane and the plate is placed on $y = 0$, $x \geqslant 0$. The inviscid shear flow is expressed as

$$U^2 = 1+2\omega_0\psi, \qquad (7.2a)$$

$$H = \tfrac{1}{2}+\omega_0\psi \qquad (7.2b)$$

and

$$e^{-\Omega} = U \qquad (7.2c)$$

in the $(\phi,\psi)$-plane. Therefore, the outer boundary condition for the viscous solution in the $(\phi,\psi^*)$-plane becomes

$$H \to \tfrac{1}{2}+\nu^{\frac{1}{2}}\omega_0\psi^*, \qquad (7.3a)$$

$$e^{-\Omega} \to [1+2\nu^{\frac{1}{2}}\omega_0\psi^*]^{\frac{1}{2}} \qquad (7.3b)$$

as $\psi^* \to \infty$.

If we change the dependent variable $H$ to $h$,

$$h = 1-2H+2\nu^{\frac{1}{2}}\omega_0\psi^*, \qquad (7.4)$$

in (2.23), we get

$$\partial h/\partial\phi = q(\partial^2 h/\partial\psi^{*2}). \qquad (7.5)$$

Here the relation between $h$ and $q$ is

$$q = [1-h+2\nu^{\frac{1}{2}}\omega_0\psi^*]^{\frac{1}{2}} \qquad (7.6)$$

and the boundary conditions are

$$h = 1 \quad \text{at} \quad \psi^* = 0, \qquad (7.7a)$$

$$h \to 0 \quad \text{as} \quad \psi^* \to \infty, \qquad (7.7b)$$

just as in the case of the uniform stream. Furthermore, changing the independent variables $\phi$ and $\psi^*$ to $\xi$,

$$\xi = 2\omega_0\sqrt{(2\nu\phi)}, \qquad (7.8)$$

and $\eta$ in (7.5), we have

$$\frac{\partial^2 h}{\partial\eta^2}-\left[\frac{1}{\eta}-\frac{1}{2}\frac{\eta^3}{q}\right]\frac{\partial h}{\partial\eta} = \xi\frac{\eta^2}{q}\frac{\partial h}{\partial\xi}, \qquad (7.9)$$

where

$$q = [1-h+\tfrac{1}{2}\xi\eta^2]^{\frac{1}{2}}. \qquad (7.10)$$

Taking $\Delta\xi = 0{\cdot}1$, we carry out the step-by-step computation up to $\xi = 3{\cdot}0$. Figure 5 shows the change of the velocity profile along the downstream boundary layer. The distribution of the skin-friction coefficient is shown in figure 6. For this problem, Glauert (1957) has obtained a perturbation solution starting with the Blasius solution, i.e.

$$C_f \Big/ \sqrt{\left(\frac{\nu}{\phi}\right)} = 0{\cdot}664 + 0{\cdot}562\xi, \tag{7.11}$$

and Mark (1962) has obtained an asymptotic solution for large $\xi$ in the form

$$C_f \Big/ \sqrt{\left(\frac{\nu}{\phi}\right)} = \frac{\xi}{\sqrt{2}}(1 + 1{\cdot}024\xi^{-\frac{4}{3}}). \tag{7.12}$$

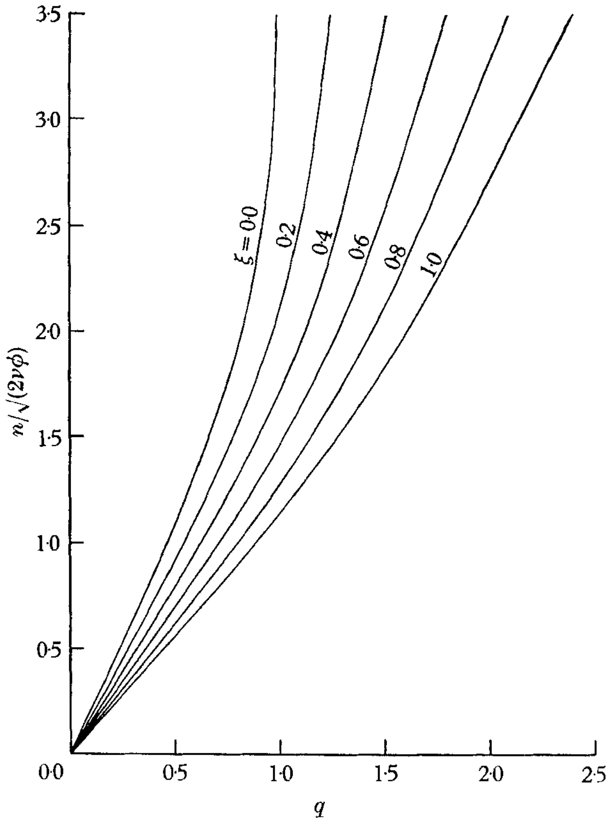


FIGURE 5. External-shear effect on the velocity profile in the boundary layer.

Their results are also included in the figure. Figure 7 shows the distribution of the displacement thickness. It should be added here that the distribution obtained by Mark by applying the momentum integral method coincides very nearly with our result.

Lastly, we must admit that the problem set up above bears a difficulty, although it may be inherent in the inverse problem. The co-ordinate of the plate surface obtained in the viscous solution is

$$y_b = -\delta^*. \tag{7.13}†$$

† Provided that the normal distance $n$ measured from the plate surface is taken in place of $y$ in (7.1), the outer inviscid flow can be expressed as

$$U = 1 + \omega_0(n - \delta^*),$$

that is, the assumption adopted by Glauert and Mark.

This fact means that the plate under consideration is not a flat plate but a hollowed-out one which accommodates the displacement thickness.

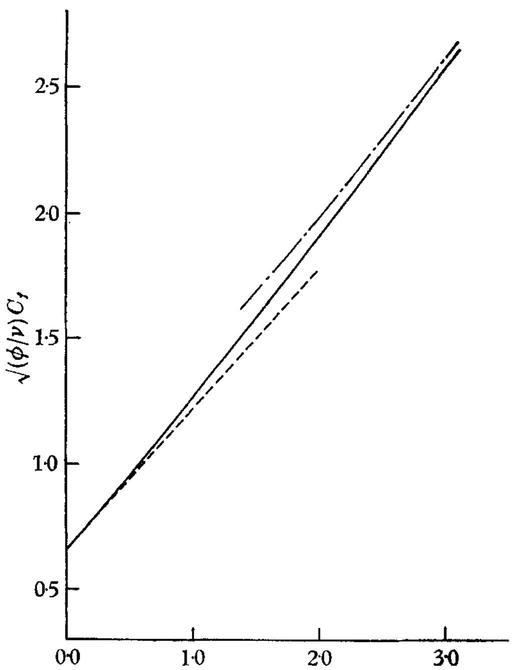


FIGURE 6

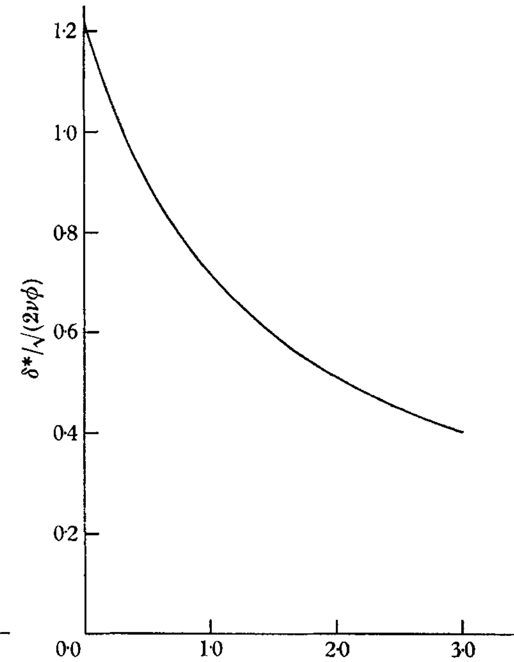


FIGURE 7

FIGURE 6. Distribution of the skin-friction coefficient in the shear flow. – – – –, Glauert (1957) perturbation solution; — – —, Mark (1962) asymptotic solution.

FIGURE 7. Distribution of the displacement thickness in the shear flow.

## 8. Slip effect

### 8.1. *Boundary condition on the wall*

In a rarefied gas flow in which the mean free path of the gas particles cannot be ignored in comparison with the boundary-layer thickness, i.e. in the so-called transition régime, the effect of slip on the boundary-layer flow must be taken into account. An accepted slip condition at the body surface is

$$q = \bar{\lambda}(\partial q/\partial n), \quad \bar{\lambda} = O(\nu), \tag{8.1}$$

although it may be somewhat fictitious for incompressible fluid flow since $\bar{\lambda}$ is proportional also to the Mach number (Murray 1965). In terms of the stretched co-ordinate $\psi^*$, the above condition is written as

$$q = \lambda(\partial q^2/\partial \psi^*), \quad \lambda = \bar{\lambda}/2\nu^{\frac{1}{2}} = O(\nu^{\frac{1}{2}}). \tag{8.2}$$

If we now ignore the curvature effect, i.e. the change of the pressure across the boundary layer, focusing our attention on the slip effect, the boundary condition to be satisfied by the function $h$ defined in (4.2) becomes

$$\sqrt{(1-h)} = -\lambda \frac{U_0}{\sqrt{(2\phi)}} \frac{\partial^2 h}{\partial \eta^2} \quad \text{at} \quad \eta = 0. \tag{8.3}$$

### 8.2. *Slip flow along a flat plate*

Let us consider a slip flow along a semi-infinite flat plate placed in a uniform stream. Change of the independent variable $\phi$ to $\xi$,

$$\xi = (1/\lambda)\sqrt{(2\phi)}, \tag{8.4}$$

in (8.3) yields

$$\partial^2 h/\partial \eta^2 = -\xi\sqrt{(1-h)} \quad \text{at} \quad \eta = 0, \tag{8.5}$$

since $U_0 = 1$. It is easy to anticipate that there arises the perfect slip at the leading edge of the plate, i.e.

$$h \equiv 0 \quad \text{at} \quad \xi = 0, \tag{8.6}$$

from the condition (8.5). Accordingly, we change the dependent variable $h$ to

$$h^* = h/\xi. \tag{8.7}$$

The differential equation for $h^*$ with respect to $(\xi, \eta)$ is

$$\frac{\partial^2 h^*}{\partial \eta^2} - \left[\frac{1}{\eta} - \frac{1}{2}\frac{\eta^3}{q}\right]\frac{\partial h^*}{\partial \eta} - \frac{\eta^2}{q} h^* = \xi \frac{\eta^2}{q}\frac{\partial h^*}{\partial \xi}, \tag{8.8a}$$

$$q = [1-\xi h^*]^{\frac{1}{2}}, \tag{8.8b}$$

and the boundary conditions are

$$\partial^2 h^*/\partial \eta^2 = -q_0 \quad \text{at} \quad \eta = 0 \tag{8.9}$$

and

$$h^* \to 0 \quad \text{as} \quad \eta \to \infty, \tag{8.10}$$

where $q_0$ is the slip velocity on the wall.

In the present problem, the boundary condition at $\eta = 0$ is not the value of $h^*$, i.e. $h_0^*$, but it is the value of its second derivative. To solve the problem, taking $h_0^*$ as an unknown value, we replace the condition (8.9) by the 5-point finite-difference formula written in such a form as

$$\sum_{n=0}^{4} a_{0,n} h_n^* = b_0 \tag{8.11}$$

and add the equation to the simultaneous equations in the form of (4.13). The coefficient $b_0$ includes $q_0$ but it can be calculated by the iteration process, i.e. by using the value of $h_0^*$ obtained in the cycle just before.

The starting equation is

$$\frac{d^2 h^*}{d\eta^2} - \left[\frac{1}{\eta} - \frac{\eta^3}{2}\right]\frac{dh^*}{d\eta} - \eta^2 h^* = 0 \tag{8.12a}$$

and

$$d^2 h^*/d\eta^2 = -1 \quad \text{at} \quad \eta = 0, \tag{8.12b}$$

$$h^* \to 0 \quad \text{as} \quad \eta \to \infty, \tag{8.12c}$$

since $q \equiv 1$ at $\xi = 0$. The solution of this linear equation, which is expressed analytically with

$$h^* = \sqrt{\left(\frac{2}{\pi}\right)} \exp\left(-\frac{t^2}{2}\right) - t\,\mathrm{erfc}\left(\frac{t}{\sqrt{2}}\right) \quad (t = \tfrac{1}{2}\eta^2) \tag{8.13}$$

(Kobayashi 1958), can be obtained without iteration. Starting with the solution, the step-by-step computation is carried out up to $\xi = 5{\cdot}0$ with $\Delta\xi = 0{\cdot}1$. The skin-friction coefficient $C_f$ is related to the slip velocity $q_0$ in the following form

$$C_f \Big/ \sqrt{\left(\frac{\nu}{\phi}\right)} = \frac{\xi}{\sqrt{2}} q_0. \tag{8.14}$$

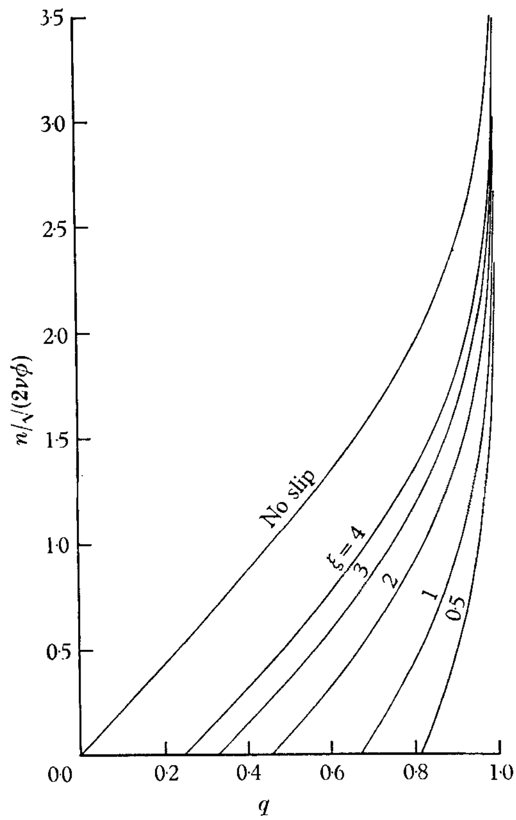


FIGURE 8. Slip effect on the velocity profile in the boundary layer (flat plate).

Figure 8 shows the variation of the velocity profile along the downstream boundary layer. The distribution of the slip velocity and of the skin-friction coefficient are shown in figures 9 and 10. An asymptotic solution for large $\xi$ has been obtained by Murray (1965), starting with the no-slip solution, i.e. the Blasius solution. His results are

$$q_0 = 0{\cdot}939/\xi + O(1/\xi^2) \tag{8.15}$$

and

$$C_f \Big/ \sqrt{\left(\frac{\nu}{\phi}\right)} = 0{\cdot}664 + O\left(\frac{1}{\xi^2}\right) \tag{8.16}$$

and those are also included in the figures. In figure 10, we find that the skin-friction coefficient increases above the no-slip value on the way. This result seems to bear a certain similarity to the one obtained by Laurmann (1961), although he treats the problem in terms of the linearized Oseen method.

The development of the displacement thickness is shown in figure 11, together with Murray's result

$$\delta^*/\sqrt{(2\nu\phi)} = 1{\cdot}217 - 2/\xi + O(1/\xi^2). \tag{8.17}$$

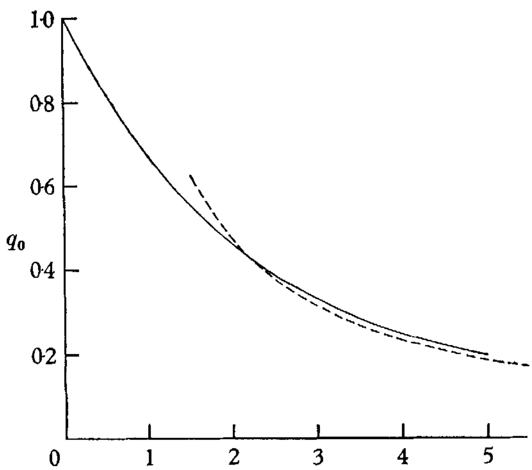


FIGURE 9

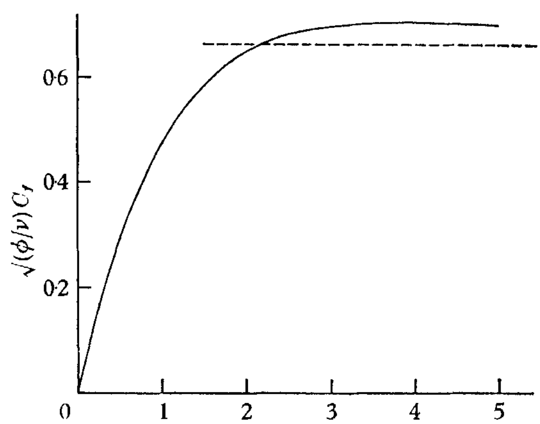


FIGURE 10

FIGURE 9. Distribution of the slip velocity on the wall (flat plate). ----, Murray (1965) asymptotic solution.

FIGURE 10. Distribution of the skin-friction coefficient in the slip flow (flat plate). ----, Murray (1965) asymptotic solution.

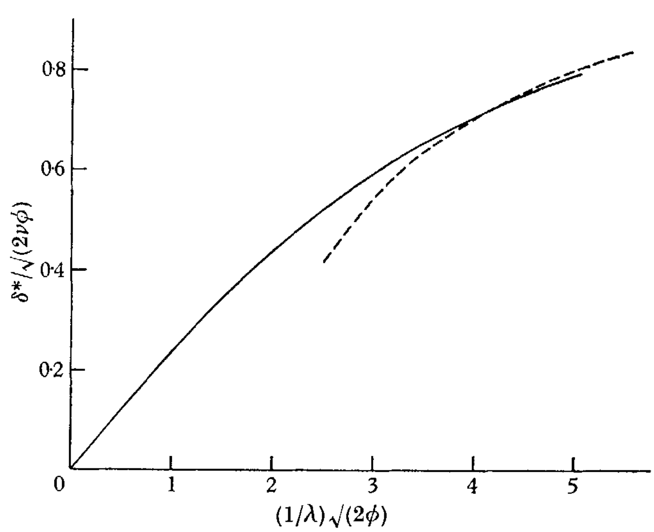


FIGURE 11. Distribution of the displacement thickness in the slip flow (flat plate). ----, Murray (1965) asymptotic solution.

In the slip flow, the increase in the boundary-layer thickness at the leading edge becomes linear, unlike the no-slip case.

### 8.3. *Slip flow around a parabolic body*

In the flat-plate problem, the slip effect is of the first-order near the leading edge. In the blunt-body problem, however, the effect remains still in the second-order near the nose point, since the boundary-layer thickness is not zero but finite at

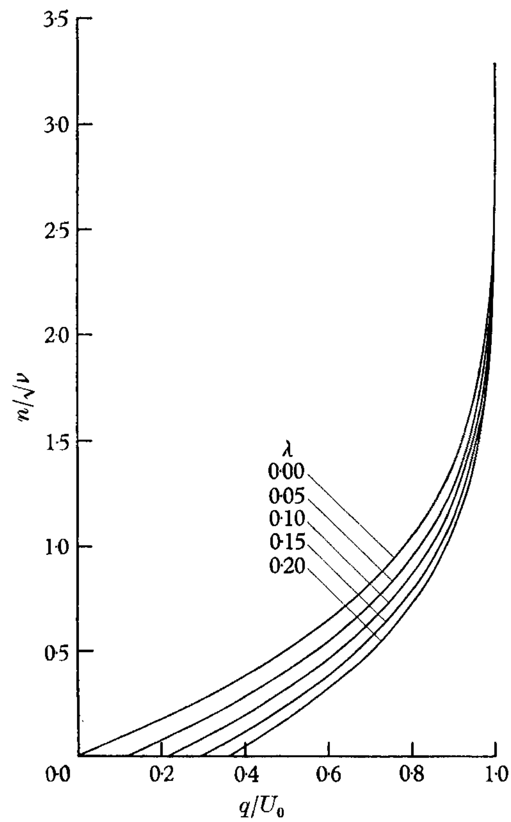


FIGURE 12. Slip effect on the velocity profile in the boundary layer at the stagnation point.

that point. In accordance with this fact, the method of solution for the blunt-body problem also becomes different from that in the case of flat plate.

Changing the variable $\phi$ to $\xi$ defined by (5.15), we can rewrite the boundary condition (8.3) as

$$\sqrt{(1-h)} = -\lambda\sqrt{(1-\xi)}\,(\partial^2 h/\partial\eta^2) \quad \text{at} \quad \eta = 0 \tag{8.18}$$

and moreover, in the form

$$h_0 = 1-\lambda^2(1-\xi)\,(\partial^2 h/\partial\eta^2)^2_{\eta=0}. \tag{8.19}$$

If we ignore the curvature effect, the problem in hand is exactly the same as the first-order problem described in § 5.2, except that the above condition should be taken in place of $h_0 = 1$. The value of $h_0$ can be obtained by the iteration process; as the value of $(\partial^2 h/\partial \eta^2)_{\eta=0}$ in (8.19), we take the value calculated by substituting the values of $h$ obtained in the cycle just before into the 5-point difference formula.

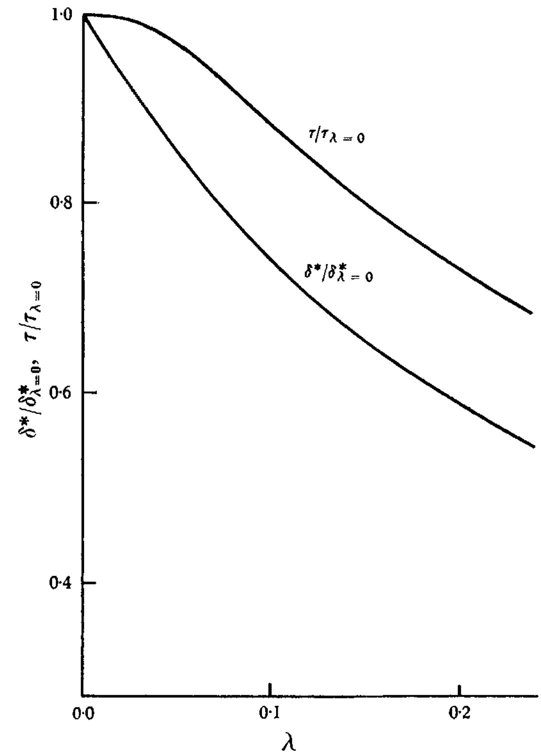


FIGURE 13. Slip effect on the skin friction and the displacement thickness at the stagnation point.

The step-by-step computations were carried out for the parameter $\lambda = 0{\cdot}05$, $0{\cdot}10$, $0{\cdot}15$ and $0{\cdot}20$, while only the results at the stagnation point are shown here. Figures 12 and 13 show the slip effect on the velocity profile in the boundary layer, the skin friction and the displacement thickness. In figure 13, where the results obtained for $\lambda = 0{\cdot}025$ and $0{\cdot}075$ are added, we find a result that the effect on the skin friction is not linear to $\lambda$. But, if we recall now that only the initial slope of the curve so obtained has in fact a significance from the analytical point of view, it should be concluded that the second-order slip effect itself on the skin friction is very little.

## 9. Conclusions

The stream function co-ordinates system is extended to viscous flow and applied to the analysis of the second-order effects on two-dimensional laminar boundary-layer flow of incompressible fluid. In the analysis, we assume that an inviscid solution in terms of the co-ordinates system is given in advance and then obtain the viscous solution to be matched to the inviscid solution within the framework of the second-order approximation. It must be recognized that such an approach is an inverse method because the body surface obtained in the viscous solution is shifted with the displacement thickness in the physical plane from the position assumed in advance. In order to clarify each effect on the flow, some practical computations are carried out for the flow along a flat plate or around a parabolic body, analytically speaking, for obtaining the viscous solution to be matched to the inviscid solution for each flow, and the results so obtained are compared with available ones obtained by the analytical methods. It should be noted that the role of the displacement thickness played in our solution is made clear through the comparison, although the restriction of the solution to the inverse problem is still to be lifted.

On reflexion, we see such a second-order problem is of real importance to the hypersonic low-density flow. Accordingly, the theory should be developed further to the flow of compressible fluids, strictly speaking, to the flow in a shock layer. In conclusion, the authors hope to have an opportunity of publishing the analysis of the hypersonic low-density flow in the near future.

The authors are indebted to Professor M. J. Lighthill, F.R.S., for his kind encouragement, and also to the referee for his helpful comments.

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